

## Chapter 7

# The General Theory of Relativity

The General Theory of Relativity is, as the name indicates, a generalization of the Special Theory of Relativity. It is certainly one of the most remarkable achievements of science to date, it was developed by Einstein with little or no experimental motivation but driven instead by philosophical questions: Why are inertial frames of reference so special? Why is it we do not feel gravity's pull when we are freely falling? Why should absolute velocities be forbidden but absolute accelerations be accepted?

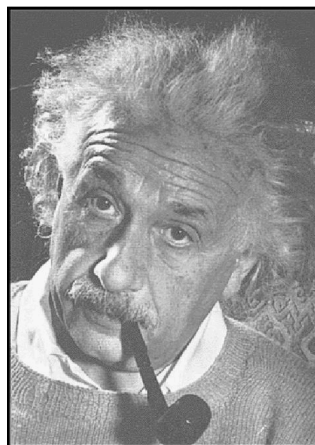


Figure 7.1: Einstein

## 7.1 The happiest thought of my life.

In 1907, only two years after the publication of his Special Theory of Relativity, Einstein wrote a paper attempting to modify Newton's theory of gravitation to fit special relativity. Was this modification necessary? Most emphatically yes! The reason lies at the heart of the Special Theory of Relativity: Newton's expression for the gravitational force between two objects depends on the masses and on the distance separating the bodies, but makes no mention of time at all. In this view of the world if one mass is moved, the other perceives the change (as a decrease or increase of the gravitational force) *instantaneously*. If exactly true this would be a physical effect which travels faster than light (in fact, at infinite speed), and would be inconsistent with the Special Theory of Relativity (see Sect. 6.2.7). The only way out of this problem is by concluding that Newton's gravitational equations are not strictly correct. As in previous occasions this does not imply that they are "wrong", it only means that they are not accurate under certain circumstances: situations where large velocities (and, as we will see, large masses) are involved cannot be described accurately by these equations.

In 1920 Einstein commented that a thought came into his mind when writing the above-mentioned paper he called it "the happiest thought of my life":

*The gravitational field has only a relative existence... Because for an observer freely falling from the roof of a house – at least in his immediate surroundings – there exists no gravitational field.*

Let's imagine the unfortunate Wile E. Coyote falling from an immense height <sup>1</sup>. As he starts falling he lets go of the bomb he was about to drop on the Road Runner way below. The bomb does not gain on Wile nor does it lag behind. If he were to push the bomb away he would see it move with constant speed in a fixed direction. This realization is important because this is exactly what an astronaut would experience in outer space, far away from all bodies (we have good evidence for this: the Apollo 10–13 spacecrafts did travel far from Earth into regions where the gravitational forces are quite weak).

Mr. Coyote is fated to repeat the experience with many other things: rocks, magnets, harpoons, anvils, etc. In all cases the same results are obtained: with respect to him all objects, irrespective of composition, mass,

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<sup>1</sup>I ignore air resistance

etc. behave as if in free space. So, if he should fall inside a closed box, he would not be able to tell whether he was plunging to his death (or, at least, severe discomfort), or whether he was in outer space on his way to Pluto at constant speed.

This is reminiscent of Galileo's argument: the observer lets go of some objects which remain in a state of uniform motion (with respect to him!). This behavior is independent of their chemical or physical nature (as above, air resistance is ignored). The observer (Wile), as long as he confines his/her observations to his/her immediate vicinity (that is, as long as he/she does not look down) has the right to interpret his state as 'at rest'. Just as Galileo argued that experiments in a closed box cannot determine the state of uniform motion of the box, Einstein argued that experiments in a freely falling small<sup>2</sup> closed box cannot be used to determine whether the box is in the grip of a gravitational force or not.

Why would this be true? The answer can be traced back to the way in which gravity affects bodies. Remember (see Sect. 4.3.3) that the quantity we called  $m$  (the mass) played two different roles in Newton's equations. One is to determine, given a force, what the acceleration of the body would be:  $F = ma$  (the inertial mass). The other is to determine the intensity with which the said body experiences a gravitational force:  $F = mMG/r^2$  (the gravitational mass). As mentioned before these two quantities need not be equal: the first "job" of  $m$  is to tell a body how much to accelerate given *any* force, a kick, an electric force (should the body be charged), etc. The second "job" tells the body how much of the gravitational force should it experience and also determines how strong a gravitational force it generates. But, in fact, both numbers are equal (to a precision of ten parts per billion).

What does this imply? Well, from Newton's equations we get

$$\frac{mMG}{r^2} = ma \quad \text{so that} \quad \frac{MG}{r^2} = a;$$

this equation determines how a body moves, which trajectory it follows, how long does it take to move from one position to another, etc. *and is independent of m!* Two bodies of different masses, composition, origin and guise will follow the same trajectory: beans, bats and boulders will move in the same way.

So the equality of the two  $m$ 's was upgraded by Einstein to a postulate: the *Principle of Equivalence*; this one statement (that the  $m$  in  $ma$  and the  $m$  in  $mMG/r^2$  are identical) implies an incredible amount of new and

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<sup>2</sup>The reasons behind the requirement that the box be small will become clear soon.

The  $m$  in  $F = ma$  is called the *inertial mass* and the  $m$  in  $mMG/r^2$  the *gravitational mass*. The inertial and gravitational masses are identical

surprising effects. The  $m$  in  $F = ma$  is called the *inertial mass* and the  $m$  in  $mMG/r^2$  the *gravitational mass*. Then the Principle of Equivalence states that the inertial and gravitational masses are identical.

The whole of the General Theory of Relativity rests on this postulate, and will fail if one can find a material for which the inertial and gravitational masses have different values. One might think that this represents a defect of the theory, its Achilles heel. In one sense this is true since a single experiment has the potential of demolishing the whole of the theory (people have tried...hard, but all experiments have validated the principle of equivalence). On the other hand one can argue that a theory which is based on a minimum of postulates is a better theory (since there are less assumptions involved in its construction); from this point of view the General Theory of Relativity is a gem <sup>3</sup>.

The completed formulation of the General Theory of Relativity was published in 1916 (Fig. 7.2).

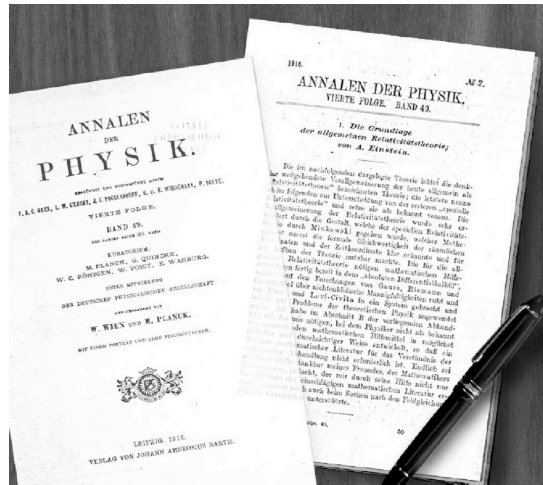


Figure 7.2: Einstein’s General Theory of Relativity paper.

<sup>3</sup>The Special Theory of Relativity is equally nice, it is based on the one statement that all inertial frames of reference are equivalent.

### 7.1.1 Newton vs. Einstein

I have stated that Newton's mechanics and his theory of gravitation are but approximations to reality and whose limitations are now known <sup>4</sup>. So it might be questionable to use  $F = ma$  and  $F_{\text{grav}} = mMG/r^2$  as basis to any argument as was done above. Einstein was careful to use these expressions only in situations where they are extremely accurate (small speeds compared to  $c$  and small gravitational forces). In these cases the inertial and gravitational masses are identical, as shown by experiment. Then he postulated that the same would be true under *all* circumstances. This statement, while consistent with Newton's equations, cannot, in a strict logical sense, be derived from them.

## 7.2 Gravitation vs acceleration

Consider the following experiment: a person is put in a room-size box high above the moon (chosen because there is no air and hence no air friction) with a bunch of measuring devices. This box is then taken high above the lunar surface and then let go: the box is then freely falling. The question is now, can the observer determine whether he/she is falling or whether he/she is in empty space unaffected by external forces (of course the answer is supposed to come before the box hits the surface). The answer to that is a definite NO! The observer can do experiments by looking at how objects move when initially at rest and when given a kick, he/she will find that they appear to move as if there were no gravitational forces at all! Similarly any experiment in physics, biology, etc. done solely inside the box will be unable to determine whether the box is freely falling or in empty space.

Why is that? Because of the equality of the gravitational and inertial masses. All objects are falling together and are assumed to be rather close to each other (the box is not immense) hence the paths they will follow will be essentially the same for each of them. So if the observer lets go of an apple, the apple and the observer follow essentially the same trajectory, and this implies that the observer will not see the apple move with respect to him. In fact, *if we accept the principle of equivalence*, nothing can be done to determine the fact that the observer is falling towards the Moon, for this can be done only if we could find some object which behaved differently from all the rest, and this can happen only if its gravitational and inertial masses

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<sup>4</sup>For all we know our present theories of mechanics and gravitation may also be invalid under certain conditions.

are different. The principle of equivalence then implies that the observer will believe that he/she is an inertial frame of reference...until disabused of the notion by the crash with the surface.

The principle of equivalence is of interest neither because its simplicity, nor because it leads to philosophically satisfying conclusions. Its importance is based on the enormous experimental evidence which confirms it; as with the Special Theory, the General Theory of Relativity is falsifiable.

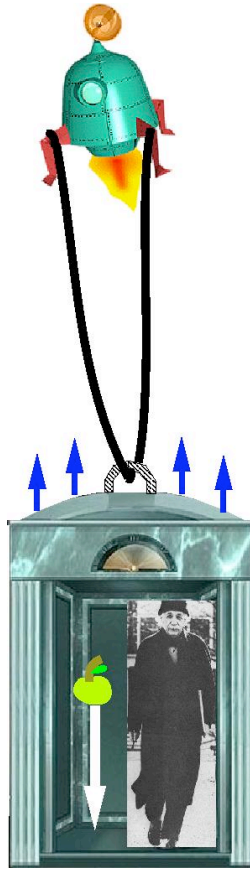


Figure 7.3: An observer cannot distinguish between acceleration produced by a rocket and the acceleration produced by gravity.

The lesson is that for any gravitational force we can always choose a frame of reference in which an observer will not experience any gravitational effects in his/her immediate vicinity (the reason for this last qualification will become clear below). Such a frame of reference is, as stated above,

For any gravitational force we can always choose a frame of reference in which an observer will not experience any gravitational effects in his/her immediate vicinity

*freely falling.*

Conversely one can take the box and attach it to a machine that accelerates it (Fig. 7.3). If an observer drops an apple in such an accelerated box he/she will see the apple drop to the floor, the observer will also feel hi/her-self pressed against the bottom of the box, etc. The observer *cannot* distinguish between this situation and the one he/she would experience in the presence of gravitational forces! As long as we do experiments in a small region, the effects produced by a gravitational force are indistinguishable from those present in an accelerated reference frame.

In a small region the effects produced by a gravitational force are indistinguishable from those present in an accelerated reference frame

Does this mean that the gravitational forces are a chimera, an illusion? Of course no. Consider for example Fig. 7.4, two apples fall to the Moon inside a box which is also falling. If they are separated by a sufficiently large distance an observer falling with the apples and box will find that the distance between the apples shortens as time goes on: this cannot be an inertial frame he argues (or else it is, but there is some force acting on the apples).

This same set-up can be used to distinguish between a box under the influence of a gravitational force and one being pulled by a machine; again we need a very big box (planet-sized). An observer places an two apples at the top of the box and releases them, he/she carefully measures its initial separation. The apples fall to the bottom of the box and the observer measures their separation there. If it is the same as above, and is the same irrespective of their initial separation, the observer is being pulled by a machine (box and all). If the separation is different, he/she can conclude that he/she is experiencing the effects of a gravitational force.

### 7.3 Light

A very surprising corollary of the above is that light paths are bent by gravitational forces! I will argue this is true in a slightly round-about way.

Consider an elevator being pulled by a crane so that it moves with constant acceleration (that is its velocity increases uniformly with time). Suppose that a laser beam propagating perpendicular to the elevator's direction of motion enters the elevator through a hole on the left wall and strikes the right wall. The idea is to compare what the crane operator and the elevator passenger see.

The crane operator, who is in an inertial frame of reference, will see the sequence of events given in Fig. 7.5. Note, that according to him/her, light travels in a straight line (as it must be since he/she is in an inertial frame!).

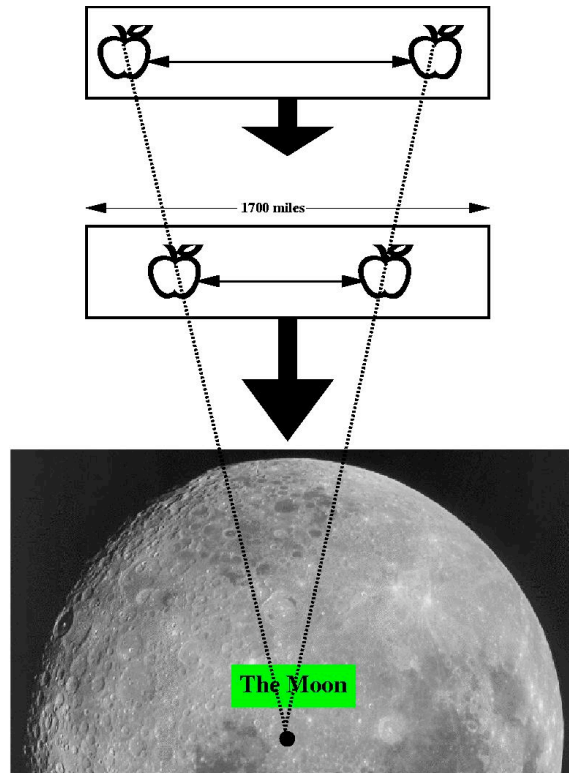


Figure 7.4: Experiment that differentiates between a gravitational effect and the effects of uniform acceleration: for an observer in the box the apples will draw closer.

The elevator passenger will see something very different as shown in Fig. 7.5: the light-path is curved! Thus for this simple thought experiment light paths will be curved for observers inside the elevator.

Now we apply the equivalence principle which implies that we cannot distinguish between an elevator accelerated by a machine and an elevator experiencing a constant gravitational force. It follows that the *same* effect should be observed if we place the elevator in the presence of a gravitational force: *light paths are curved by gravity*

Light light paths are curved by gravity

That gravity affects the paths of planets, satellites, etc. is not something strange. But we tend to think of light as being different somehow. The above argument shows that light is not so different from other things and is indeed affected by gravity in a very mundane manner (the same elevator experiment could be done by looking at a ball instead of a beam of light and the same



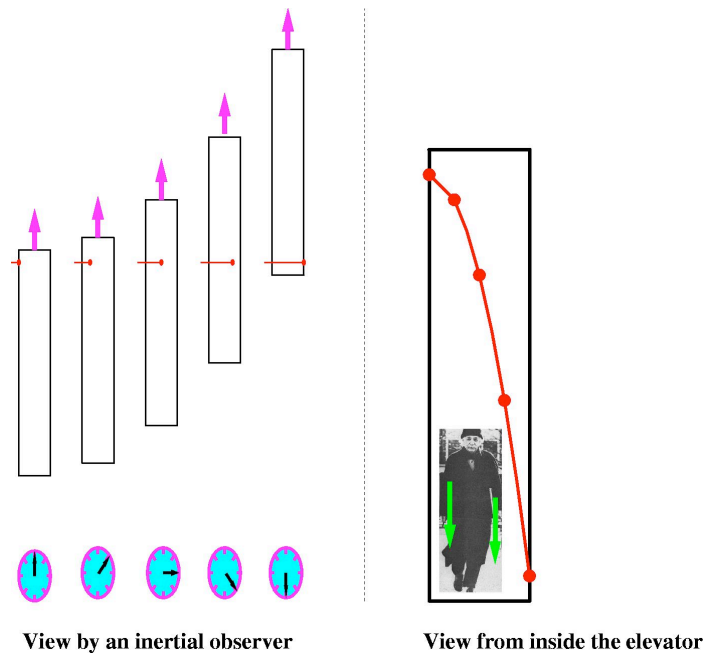


Figure 7.5: **Left:** sequence of events seen by a crane operator lifting an elevator at constant acceleration (the speed increases uniformly with time). The short horizontal line indicates a laser pulse which, at the initial time, enters through an opening on the left-hand side of the elevator. At the final time the light beam hits the back wall of the elevator. **Right:** same sequence of events seen by a passenger in an elevator being hoisted by a crane. The line joining the dots indicates the path of a laser which, at the initial time, enters through an opening on the left-hand side of the elevator. At the final time the light beam hits the back wall of the elevator.

sort of picture would result).

A natural question is then, why do we not see light fall when we ride an elevator? The answer is that the effect in ordinary life is very small. Suppose that the height of the elevator in Fig. 7.5 is 8 ft. and its width is 5 ft; if the upward acceleration is 25% that of gravity on Earth then the distance light falls is less than a millionth of the radius of a hydrogen atom (the smallest of the atoms). For the dramatic effect shown in the figure the acceleration must be enormous, more than  $10^{16}$  times the acceleration of gravity on Earth (this implies that the passenger, who weights 70 kg on Earth, will weigh more than 1,000 trillion tons in the elevator).

This does not mean, however, that this effect is completely unobservable (it is small for the case of the elevator because elevators are designed for very small accelerations, but one can imagine other situations). Consider from example a beam of light coming from a distant star towards Earth (Fig. 7.6) which along the way comes close to a very massive dark object. The arguments above require the light beam to bend; and the same thing will happen for any other beam originating in the distant star. Suppose that the star and the opaque object are both perfect spheres, then an astronomer on Earth will see, not the original star, but a *ring* of stars (often called an “Einstein ring”). If either the star or the massive dark object are not perfect spheres then an astronomer would see several images instead of a ring (Fig. 7.7). This effect has been christened *gravitational lensing* since gravity acts here as a lens making light beams converge.

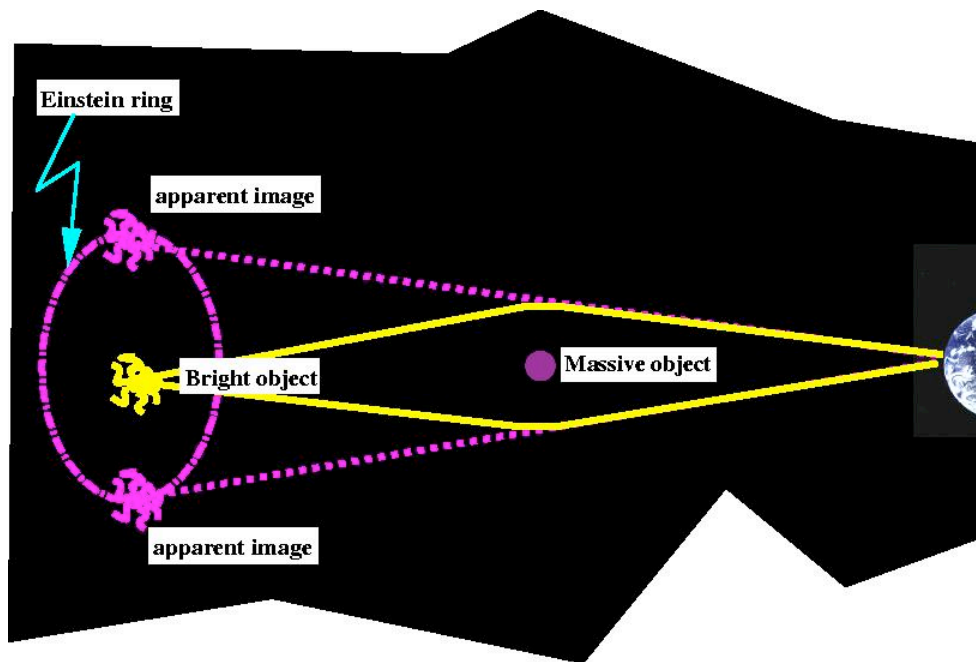


Figure 7.6: Diagram illustrating the bending of light from a star by a massive compact object. If both the bright objects and the massive object are perfect sphere, there will be an apparent image for every point on the “Einstein ring”.

How do we know that the multiple images which are sometimes seen (Fig. 7.7) are a result of the bending of light? The argument is by contradiction:

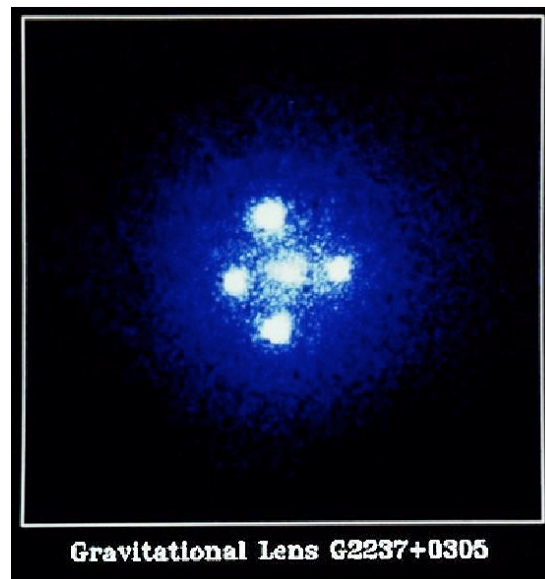


Figure 7.7: The Einstein Cross: four images of a quasar GR2237+0305 (a very distant – 8 billion light-years–, very bright object) appear around the central glow. The splitting of the central image is due to the gravitational lensing effect produced by a nearby galaxy. The central image is visible because the galaxy does not lie on a straight line from the quasar to Earth. The Einstein Cross is only visible from the southern hemisphere.

suppose they are not, that is suppose, that the images we see correspond to different stars. Using standard astronomical tools one can estimate the distance between these stars; it is found that they are separated by thousands of light years, yet it is observed that if one of the stars change, all the others exhibit the same change instantaneously! Being so far apart precludes the possibility of communication between them; the simplest explanation is the one provided by the bending of light. It is, of course, possible to ascribe these correlations as results of coincidences, but, since these correlations are observed in many images, one would have to invoke a “coincidence” for hundreds of observations in different parts of the universe.

The bending of light was one of the most dramatic predictions of the General Theory of Relativity, it was one of the first predictions that were verified as we will discuss below in Sect. 7.12.

## 7.4 Clocks in a gravitational force.

When comparing a clock under the influence of gravitational forces with one very far from such influences it is found that the first clock is slow compared to the second. To see this consider the same clock we used in the Special Theory of Relativity. For this experiment, however, imagine that the clock is being accelerated upward, being pulled by a crane. The clock gives off a short light pulse which moves towards the mirror at the top of the box, at the same time the mirror recedes from the pulse with even increasing speed (since the box accelerates). Still the pulse eventually gets to the mirror where it is reflected, now it travels downward where the floor of the box is moving up also with ever increasing velocity (see Fig. 7.8).

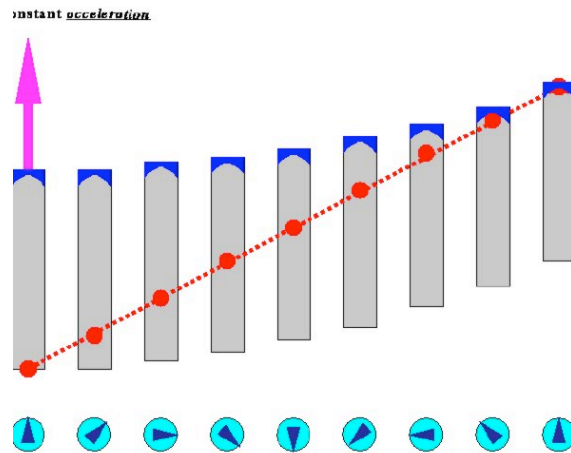


Figure 7.8: An accelerated clock. The circle denotes a pulse of light which at the initial is sent from a source; after a time it reaches the top of the the box and is reflected. The time it takes to do the trip is *longer* than for a clock at rest.

On the trip up the distance covered by light is larger than the height

of the box at rest, on the trip down the distance is smaller. A calculation shows that the whole distance covered in the trip by the pulse is larger than twice the height of the box, which is the distance covered by a light pulse when the clock is at rest.

Since light always travels at the same speed, it follows that the time it takes for the pulse to go the round trip is longer when accelerating than when at rest: *clocks slow down whenever gravitational forces are present.*

This has an amazing consequence: imagine a laser on the surface of a very massive and compact planet (so that the gravitational field is very strong). An experimenter on the planet times the interval between two crests of the laser light waves and gets, say, a millionth of a second. His clock, however, is slow with respect to the clock of an observer far away in deep space, this observer will find that the time between two crests is larger. This implies that the frequency of the laser is larger on the planet than in deep space: *light leaving a region where gravity is strong reddens.* This is called the *gravitational red-shift* (see Fig. 7.9).

Light leaving a region where gravity is strong reddens

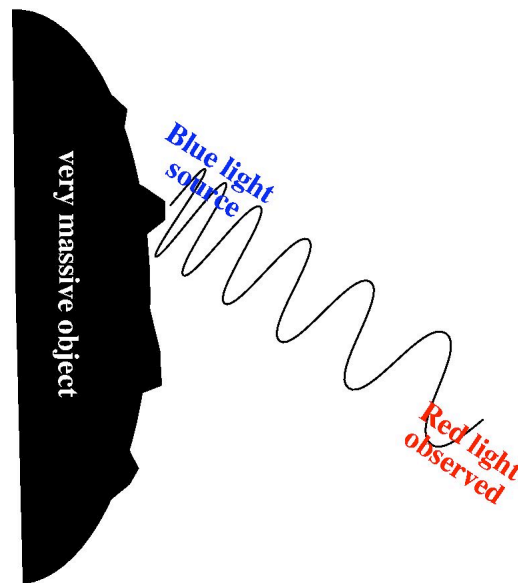


Figure 7.9: The gravitational redshift. Since clocks slow down in a strong gravitational field then light, whose oscillations can be used as clocks, will be shifted towards the red as it leaves a region where gravity is strong.

As for time dilation, the slowing down of clocks in the presence of gravitational forces affects *all* clocks, including biological ones. A twin traveling to

a region where gravity is very strong will come back a younger than the twin left in a rocket in empty space. This is an effect on top of the one produced by time dilation due to the motion of the clocks. The gravitational forces required for a sizable effect, however, are enormous. So the twin will return younger...provided she survives.

## 7.5 Black holes

So gravity pulls on light just as on rocks. We also know that we can put rocks in orbit, can we put light in orbit? Yes! but we need a very heavy object whose radius is very small, for example, we need something as heavy as the sun but squashed to a radius of less than about 3km. Given such an object, light moving towards it in the right direction will, if it comes close enough land in an orbit around it. If you place yourself in the path of light as it orbits the object, you'd be able to see your back.

But we can go farther and imagine an object so massive and compact that if we turn on a laser beam on its surface gravity's pull will bend it back towards the surface. Think what this means: since no light can leave this object it will appear perfectly black, this is a *black hole*. An object which comes sufficiently close to a black hole will also disappear into it (since nothing moves faster than light if an object traps light it will also trap everything else).

The effect of a black holes, like all gravitational effects, decreases with distance. This means that there will be a "boundary" surrounding the black hole such that anything crossing it will be unable to leave the region near the black hole; this boundary is called the *black-hole horizon* see Fig. 7.10 Anything crossing the horizon is permanently trapped. Black holes are perfect roach motels: once you check in (by crossing the horizon), you never check out.

The distance from the black hole to the horizon is determined by the mass of the black hole: the larger the mass the more distant is the horizon from the center. For a black hole with the same mass as our sun the horizon is about 3 km from the center; for black holes with a billion solar masses (yes there *are* such things) this is increased to  $3 \times 10^9$  km, about the distance from the sun to Uranus. For very massive black holes the horizon is so far away from the center that an observer crossing it might not realize what has just happened, only later, when all efforts to leave the area prove futile, the dreadful realization of what happened will set in.

Imagine a brave (dumb?) astronaut who decides to through the horizon

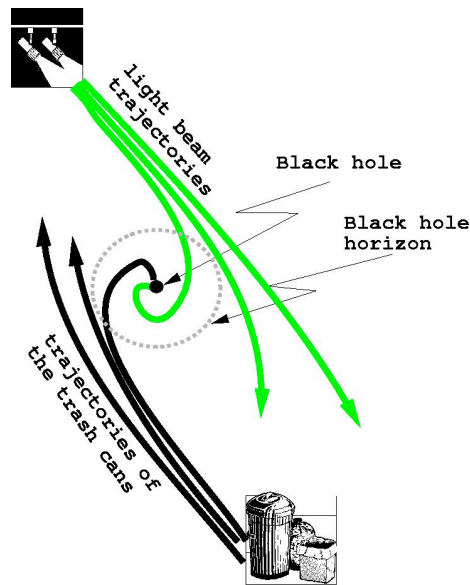


Figure 7.10: Illustration of the horizon surrounding the black hole. The black hole is represented by the small heavy dot, the light rays or particle trajectories which cross the dotted line cannot cross it again.

and into the nearest black hole and let us follow his observations. The first effects that becomes noticeable as he approaches the event horizon is that his clock ticks slower and slower with respect to the clocks on his spaceship very far from the black hole (see Sect. 7.4) to the point that it will take infinite spaceship time for him to cross the horizon. In contrast it will take a finite amount of astronaut time to cross the horizon, an extreme case of the relativity of time.

As the astronaut approaches the horizon the light he emits will be more and more shifted towards the red (see Sect. 7.4) eventually reaching the infrared, then microwaves, then radio, etc. In order to see him the spaceship will eventually have to detect first infrared light, then radio waves, then microwaves, etc.

After crossing the horizon the astronaut stays inside. Even though the crossing of the horizon might not be a traumatic experience the same cannot be said for his ultimate fate. Suppose he decides to fall feet first, then, when sufficiently close to the black hole, the gravitational pull on his feet will be much larger than that on his head and he will be literally ripped to pieces.

So far black holes appear an unfalsifiable conclusion of the General The-

ory of Relativity: their properties are such that no radiation comes out of them so they cannot be detected from a distance, and if you should decide to go, you cannot come back to tell your pals whether it really was a black hole or whether you died in a freak accident. Doesn't this contradict the basic requirement that a scientific theory be falsifiable (Sect. 1.2.1)?

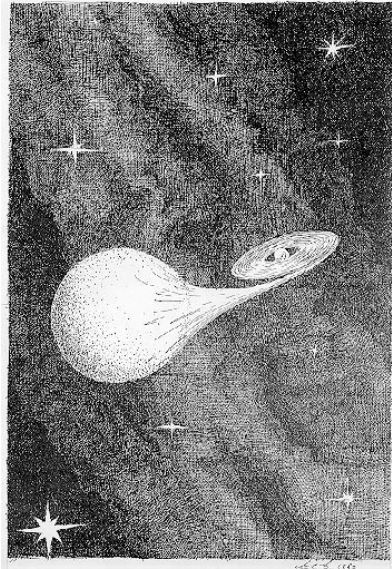


Figure 7.11: Artist's version of a black hole accreting matter from a companion star. The Star is on the left of the picture and is significantly deformed by the gravitational pull of the black hole; the object on the right represents the matter which surrounds the black hole and which is being sucked into it. The black hole is too small to be seen on the scale of this picture

Well, no, General Theory of Relativity even in this one of its most extreme predictions *is* falsifiable. The saving circumstance is provided by the matter surrounding the black hole. All such stuff is continuously being dragged into the hole (see Fig. 7.11) and devoured, but in the process it gets extremely hot and radiates light, ultraviolet radiation and X rays. Moreover, this cosmic Maelstrom is so chaotic that the radiation changes very rapidly, sometimes very intense, sometimes much weaker, and these changes come very rapidly (see Fig. 7.12). From this changes one can estimate the size of the object generating the radiation.

On the other hand astronomers can see the gravitational effects on nearby stars of whatever is making the radiation. And from these effects they



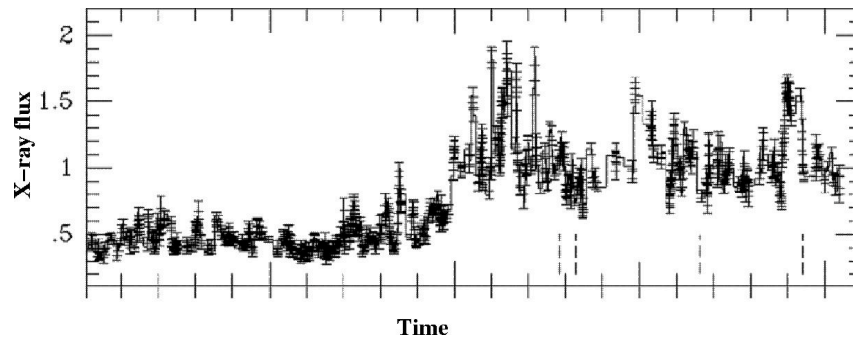


Figure 7.12: X-ray emission from a black hole candidate (Cygnus X1)

can estimate the mass of the beast. Knowing then the size, the manner in which matter radiates when it comes near, and the mass one can compare this to the predictions of General Theory of Relativity and decide whether this is a black hole or not. The best candidate for a black hole found in this way is called Cygnus X1 (the first observed X ray source in the constellation Cygnus, the swan).

All the ways we have of detecting black holes depend on the manner in which they affect the matter surrounding them. The most striking example is provided by some observation of very distant X-ray sources which are known to be relatively compact (galaxy size) and very far away. Then the very fact that we can see them implies that they are extremely bright objects, so bright that we know of only one source that can fuel them: the radiation given off by matter while being swallowed by a black hole <sup>5</sup>. So the picture we have of these objects, generically called active galactic nuclei, is that of a supermassive (a billion solar masses or so) black hole assimilating many stars per second, and in disappearing these stars give off the energy that announces their demise.

All this from the (apparently) innocent principle of equivalence.

## 7.6 Gravitation and energy

Consider a beam of sunlight falling on your skin; after a while your skin warms and, eventually, will burn: light carries energy (which is absorbed by your skin thus increasing its temperature). Recall also that a body with

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<sup>5</sup>This is much more efficient than nuclear power which would be incapable of driving such bright sources.

mass  $m$ , by its very existence, carries an energy  $mc^2$  (Sec. 6.2.8). There is no way, however, in which we can associate a mass with light; for example, we can always change the speed of a mass (even if only a little bit), but this cannot be done with light.

Gravity will affect anything carrying energy

The force of gravity affects both light and all material bodies; since both carry energy, but only the bodies carry mass, it follows that gravity *will affect anything carrying energy*. This conclusion lies at the root of the construction of Einstein's equations which describe gravity.

Note that this conclusion has some rather strange consequences. Consider for example a satellite in orbit around the Earth, when the Sun shines on it it will increase its energy (it warms up), and gravity's pull with it. When the satellite is in darkness it will radiate heat, lose energy and the force of gravity on it will decrease <sup>6</sup>.

Again let me emphasize that this argument is *not* intended to imply that light carries mass, but that gravity will affect anything that carries energy.

## 7.7 Space and time.

When considering the Special Theory of Relativity we concluded that the state of motion of an observer with respect to, say, a laboratory, determines the rate at which his/her clocks tick with respect to the lab's clocks (see Sect.6.2.3). Thus, in this sense, time and space mingle: the position of the observer (with respect to the lab's measuring devices) determines, as time evolves, his/her state of motion, and this in turn determines the rate at which his/her clocks tick with respect to the lab's.

Now consider what happens to objects moving under the influence of a gravitational force: if initially the objects set out at the same spot with the same speed they will follow the same path (as required by the principle of equivalence). So what!? To see what conclusions can be obtained let me draw a parallel, using another murder mystery.

Suppose there is a closed room and a line of people waiting to go in. The first person goes in and precisely two minutes afterward, is expelled through a back door, dead; it is determined that he died of a blow to the head. The police concedes that the room is worth investigating, but procrastinates, alleging that the person was probably careless and his death was accidental. Soon after, however, a second victim enters the room with precisely the same results, she also dies of an identical blow to the head; the police claims an astounding coincidence: two accidental deaths. This goes on for many

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<sup>6</sup>Needless to say this is a very small effect, of the order of one part in a trillion.

hours, each time the victim dies of the same thing irrespective of his/her age, occupation, habits, color, political persuasion or taste in Pepsi vs. Coke; animals suffer the same fate, being insects or whales. If a rock is sent flying in, it comes out with a dent of the same characteristics as the ones suffered by the people and animals.

The police finally shrewdly concludes that there is something in the room that is killing people, they go in and... But the result is not important, what is important for this course is the following. We have a room containing something which inflicts a certain kind of blow to everything going through the room, I can then say that this inflicting of blows is a property of the room.

Consider now a region of empty space relatively near some stars. Assume that the only force felt in this region is the gravitational pull of these stars, hence all objects, people, animals, etc. going into this region will accelerate in precisely the same way. Then I can state that the region in space has a property which generates this acceleration <sup>7</sup>.

Remember however that the region considered was in empty space (it only contains the objects we send into it), yet some property of this region determines the motion of anything that goes through it; moreover this property is a result of the gravitational pull of nearby heavy objects. The conclusion is then that *gravity alters the properties of space*, we also saw that the rates of clocks are altered under the influence of a gravitational force, it follows that *gravity alters the properties of space and time*. Space and time is in fact very far from the unchanging arena envisaged by Newton, they are dynamical objects whose properties are affected by matter and energy. These changes or deformations of space and time in turn determine the subsequent motion of the bodies in space time: matter tells space-time how to curve and space-time tells matter how to move (Fig. 7.13).

Gravity alters the properties of space and time

Matter tells space-time how to curve and space-time tells matter how to move

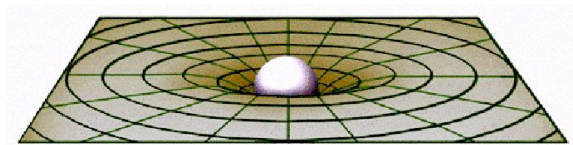


Figure 7.13: An illustration of the bending of space produced by a massive object

<sup>7</sup>I assume that the objects coming into this region are not too heavy, so that their gravitational forces can be ignored and that they start from the same spot with identical velocities.

## 7.8 Properties of space and time.

Up to here I've talked little of the implications of the Special Theory of Relativity on the General Theory of Relativity, I have only argued that in special relativity time and space are interconnected. In a separate discussion I argued that gravity alters space. In this section I will use what we know about length contraction together with the equivalence principle to determine how space is altered by gravity and to show that it is this deformation of space that is responsible for the gravitational force.

Imagine two disks, one of which is made to rotate uniformly as in 7.14; each disk has its own observer provided with a meter stick, labeled  $\ell$  and  $\ell_o$  in the figure. The disks are so constructed that when overlapping their circumferences match. The rotating meter stick is continually moving along its length so that  $\ell$  will be length-contracted with respect to  $\ell_o$ , so a larger number of  $\ell$  will fit in the circumference. This means that the rotating observer measures a longer circumference than the non-rotating (inertial) observer.

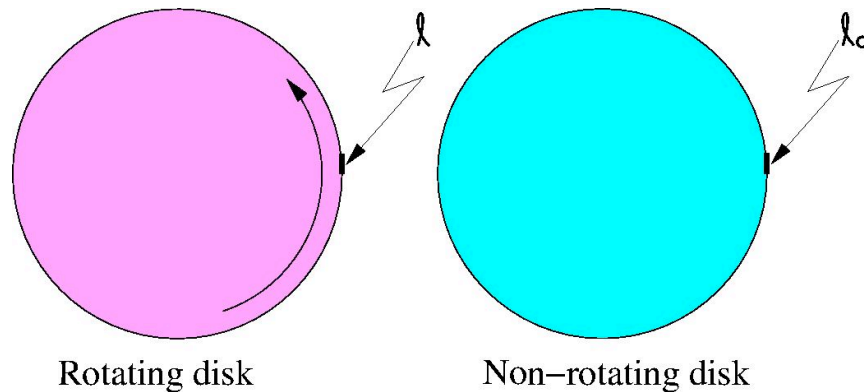


Figure 7.14: A rotating vs a non-rotating disk. The bit labeled  $\ell$  in the rotating disk is shorter, due to length contraction to the corresponding bit  $\ell_o$  in the non-rotating disk.

Consider now a radius of the disks. This is a length that is always perpendicular to the velocity of the disk and it is unaffected by the rotation: both disks will continue to have the same radius (see Sect. 6.2.4).

So now we have one non-rotating disk whose circumference is related to the radius by the usual formula,  $circumference = 2\pi \times radius$ , and a rotating disk whose observer measures a *larger* circumference but the *same* radius. In the rotating observer the formula does not hold!

How can this be? Isn't it true that the perimeter always equals  $2\pi$  radius? The answer to the last question is yes...provided you draw the circle on a flat sheet of paper. Suppose however that you are constrained to draw circles on a sphere, and that you are forced to measure distances only on the sphere. Then you find that the perimeter *measured along the sphere* is smaller than  $2\pi \times \text{radius}$  (with the radius also measured along the sphere, see Fig. 7.15).

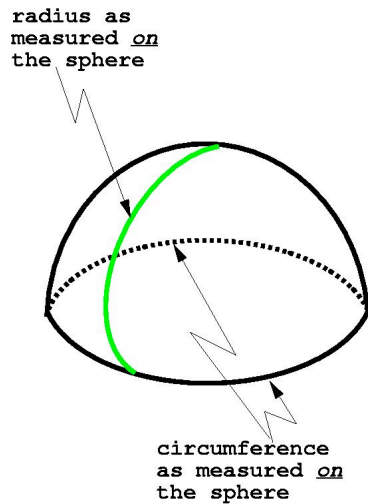


Figure 7.15: The distance from the equator to the pole on a sphere is larger than the radius. For being constrained to move on the surface of the sphere this distance is what *they* would call the radius of their universe, thus for them the circumference is smaller than  $2\pi \times \text{radius}$  and they can conclude that they live in curved space.

A similar situation is observed in the rotating disk with a similar solution: the reason the rotating observer finds that the circumference is not equal to  $2\pi$  times the radius is that this observer is in a curved surface. On a sphere we just saw that  $2\pi \times > \text{circumference}$ , in a saddle-shaped surface  $2\pi \times < \text{circumference}$  as in the situation we have been looking at.

We conclude that the uniformly rotating disk behaves as a (piece of a) saddle-shaped due to length contraction. So much for the effects of special relativity.

Now let us go back to the principle of equivalence. One of its consequences is that, by doing experiments in a small region one cannot distinguish between a gravitational force and an accelerated system. So if we

attach a small laboratory of length  $\ell_0$  (at rest) to the small section of the perimeter, experiments done there will not be able to tell whether the lab. is in a rotating disk or experiences a gravitational force (remember that a rotating object is changing its velocity – in direction – and it is therefore accelerating!).

Putting together the above two arguments we get

*Gravitation curves space and time.*

Conversely curved space and time generate effects which are equivalent to gravitational effects. In order to visualize this imagine a world where all things can only move on the surface of a sphere. Consider two beings labeled **A** and **B** as in Fig. 7.16, which are fated live on the surface of this sphere. On a bright morning they both start from the equator moving in a direction perpendicular to it (that is, they don't meander about but follow a line perpendicular to the equator).

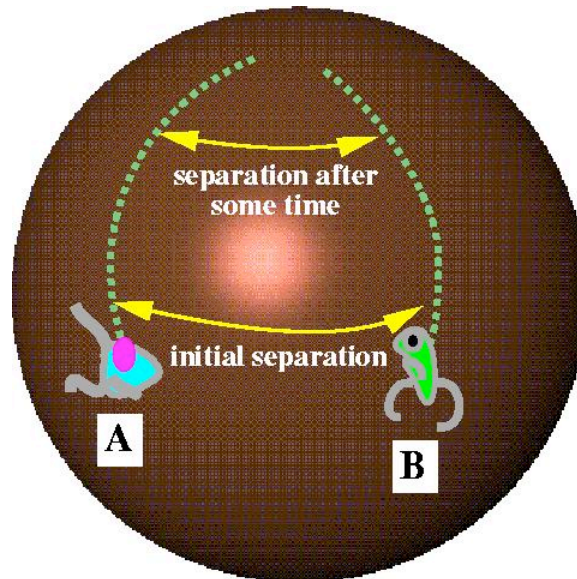


Figure 7.16: Two beings moving on a sphere are bound to come closer just as they would under the effects of gravity

As time goes on the two beings will come closer and closer. This effect is similar to the experiment done with two apples falling towards the moon (Fig. 7.4): an observer falling with them will find their distance decreases as time progresses; sentient apples would find that they come closer as time goes on.

So we have two descriptions of the same effect: on the one hand gravitational forces make the apples approach each other; on the other hand the fact that a sphere is curved makes the two beings approach each other; mathematically both effects are, in fact, identical. In view of this the conclusion that gravity curves space might not be so peculiar after all; moreover, in this picture the equivalence principle is very natural: bodies move the way they do due to the way in which space is curved and so the motion is independent of their characteristics <sup>8</sup>, in particular the mass of the body does not affect its motion.

Bodies move the way they do due to the way in which space is curved and so the motion is independent of their characteristics

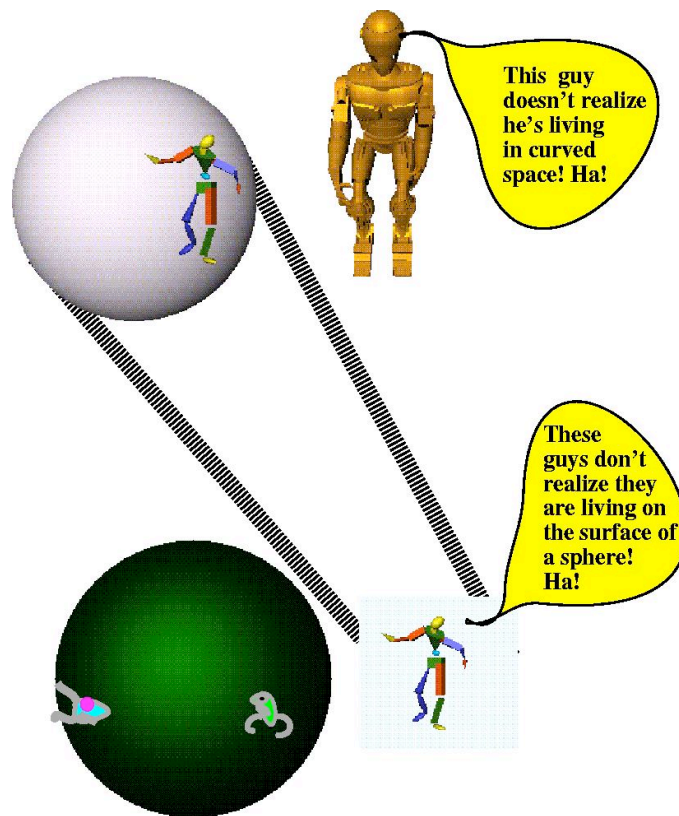


Figure 7.17: Just as bugs fated to live on the surface of a sphere might find it peculiar to learn their world is curved, so we might find it hard to realize that our space is also curved.

Now the big step is to accept that the same thing that happened to the

<sup>8</sup>I am assuming here that the moving things are not massive enough to noticeably curve space on their own.

above beings is happening to us all the time. So how come we don't see that the space around us is really curved? The answer is gotten by going back to the beings **A** and **B**: they cannot "look out" away from the sphere where they live, they have no perception of the perpendicular dimension to this sphere, and so they cannot "see it from outside" and realize it is curved. The same thing happens to us, we are inside space, in order to see it curved we would have to imagine our space in a larger space of more dimensions and *then* we could see that space is curved; Fig. 7.17 gives a cartoon version of this.

## 7.9 Curvature

When considering the beings living on a sphere it is easy for us to differentiate between the sphere and some plane surface: we actually see the sphere being curved. But when it comes to us, and our curved space, we cannot see it since this would entail our standing outside space and looking down on it. Can we then determine whether space is curved by doing measurements inside it?

To see that this can be done let's go back to the beings on the sphere. Suppose they make a triangle by the following procedure: they go from the equator to the north pole along a great circle (or meridian) of the sphere, at the north pole they turn  $90^\circ$  to the right and go down another great circle until they get to the equator, then they make another  $90^\circ$  turn to the right until they get to the starting point (see Fig. 7.18). They find that all three lines make  $90^\circ$  angles with each other, so that the sum of the angles of this triangle is  $270^\circ$ , knowing that angles in all flat triangles always add up to  $180^\circ$  they conclude that the world they live on is not a flat one. Pythagoras' theorem only holds on flat surfaces

We can do the same thing: by measuring very carefully angles and distances we can determine whether a certain region of space is curved or not. In general the curvature is very slight and so the distances we need to cover to observe it are quite impractical (several light years), still there are some special cases where the curvature of space is observed: if space were flat light would travel in straight lines, but we observe that light does no such thing in regions where the gravitational forces are large; I will discuss this further when we get to the tests of the General Theory of Relativity in the following sections.

The curvature of space is real and is generated by the mass of the bodies in it. Correspondingly the curvature of space determines the trajectories of



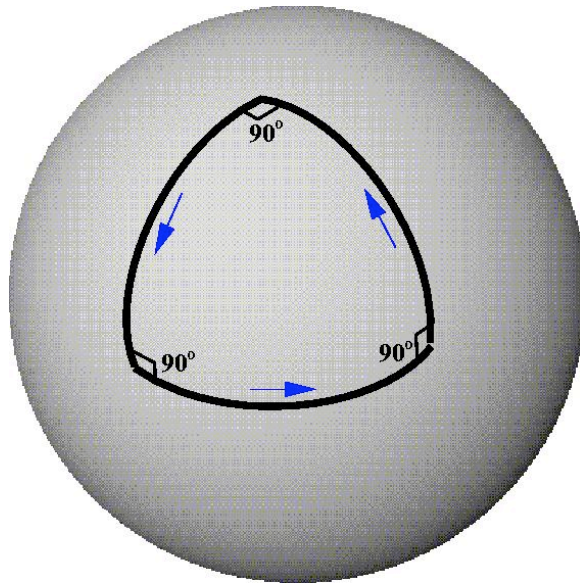


Figure 7.18: A path followed by a determined being living on the surface of a sphere; each turn is at right angles to the previous direction, the sum of the angles in this triangle is then  $270^\circ$  indicating that the surface in which the bug lives is not flat.

all bodies moving in it. The Einstein equations are the mathematical embodiment of this idea. Their solutions predict, given the initial positions and velocities of all bodies, their future relative positions and velocities. In the limit where the energies are not too large and when the velocities are significantly below  $c$  the predictions of Einstein's equations are indistinguishable from those obtained using Newton's theory. At large speeds and/or energies significant deviations occur, and Einstein's theory, not Newton's, describes the observations.

## 7.10 Waves

A classical way of picturing the manner in which heavy bodies curve space is to imagine a rubber sheet. When a small metal ball is made to roll on it it will go in a straight line at constant speed (neglecting friction). Now imagine that a heavy metal ball is placed in the middle of the sheet; because of its weight the sheet will be depressed in the middle (Fig 7.13). When a small ball is set rolling it will no longer follow a straight line, its path will

be curved and, in fact, it will tend to circle the depression made by the heavy ball. The small ball can even be made to orbit the heavy one (it will eventually spiral in and hit the heavy ball, but that is due to friction, if the sheet is well oiled it takes a long time for it to happen). This toy then realizes what was said above: a heavy mass distorts space (just as the heavy ball distorts the rubber sheet). Any body moving through space experiences this distortion and reacts accordingly.

Now imagine what happens if we drop a ball in the middle of the sheet. It will send out ripples which spread out and gradually decrease in strength. Could something similar happen in real life? The answer is yes! When there is a rapid change in a system of heavy bodies a large amount of gravitational waves are produced. These waves are ripples in space which spread out from their source at the speed of light carrying energy away with them.

A computer simulation of a gravitational wave is given in Fig. 7.19. The big troughs denote regions where the wave is very intense, the black dot at the center denotes a black hole, the ring around the hole represents the black hole's horizon.



Figure 7.19: A computer simulation of a gravitational wave generated by a collision of two black holes, which have now merged and are represented by the heavy black dot in the middle.

Can we see gravitational waves? Not yet directly, but we have very strong indirect evidence of their effects. Several systems which according to the General Theory of Relativity ought to lose energy by giving off gravitational waves have been observed. The observations show that these systems lose energy, and the rate at which this happens coincides precisely with the

predictions from the theory.

Observing gravitational waves directly requires very precise experiments. The reason is that, as one gets farther and farther away from the source these waves decrease in strength very rapidly. Still, if a relatively strong gravitational wave were to go by, say, a metal rod, its shape would be deformed by being stretched and lengthened periodically for a certain time. By accurately measuring the length of rods we can hope to detect these changes. The technical problems, however, are enormous: the expected variation is of a fraction of the size of an atom! Nonetheless experiments are under way.

Gravitational waves are generated appreciably only in the most violent of cosmic events. During the last stages in the life of a star heavier than 3 solar masses, most of the stellar material collapses violently and inexorably to form a black hole (in the rubber sheet picture this corresponds to dropping a very small and very heavy object on the sheet). The corresponding deformation of space travels forth from this site as a gravitational wave. High intensity gravitational waves are also produced during the collision of two black holes or any sufficiently massive compact objects.

### 7.11 Summary.

The conclusions to be drawn from all these arguments are,

- *All* frames of reference are equivalent, provided we are willing to include possible gravitational effects (in non-inertial or accelerated frames forces will appear which are indistinguishable from gravitational forces).
- Space-time is a dynamic object: matter curves it, and the way in which it is curved determines the motion of matter in it. Since all bodies are affected in the same way by the curvature of space and time the effects of gravity are independent of the nature of the body. Changes in the distribution of matter change space-time deforming it, and, in some instances, making it oscillate.

### 7.12 Tests of general relativity.

After Einstein first published the General Theory of Relativity there was a very strong drive to test its consequences; Einstein himself used his equations to explain a tiny discrepancy in the motion of Mercury. Yet the most dramatic effect was the shifting of the positions of the stars (see below).

Since 1916 there have been many measurements which agree with the General Theory of Relativity to the available accuracy. Here I will concentrate on the “classical” tests of the theory.

### 7.12.1 Precession of the perihelion of Mercury

A long-standing problem in the study of the Solar System was that the orbit of Mercury did not behave as required by Newton’s equations.

To understand what the problem is let me describe the way Mercury’s orbit looks. As it orbits the Sun, this planet follows an ellipse...but only approximately: it is found that the point of closest approach of Mercury to the sun does not always occur at the same place but that it slowly moves around the sun (see Fig. 7.20). This rotation of the orbit is called a *precession*.

The precession of the orbit is not peculiar to Mercury, *all* the planetary orbits precess. In fact, Newton’s theory predicts these effects, as being produced by the pull of the planets on one another. The question is whether Newton’s predictions agree with the *amount* an orbit precesses; it is not enough to understand qualitatively what is the origin of an effect, such arguments must be backed by hard numbers to give them credence. The precession of the orbits of all planets *except* for Mercury’s can, in fact, be understood using Newton’s equations. But Mercury seemed to be an exception.

As seen from Earth the precession of Mercury’s orbit is measured to be 5600 seconds of arc per century (one second of arc =  $1/3600$  degrees). Newton’s equations, taking into account all the effects from the other planets (as well as a very slight deformation of the sun due to its rotation) and the fact that the Earth is not an inertial frame of reference, predicts a precession of 5557 seconds of arc per century. There is a discrepancy of 43 seconds of arc per century.

This discrepancy cannot be accounted for using Newton’s formalism. Many *ad-hoc* fixes were devised (such as assuming there was a certain amount of dust between the Sun and Mercury) but none were consistent with other observations (for example, no evidence of dust was found when the region between Mercury and the Sun was carefully scrutinized). In contrast, Einstein was able to *predict*, without any adjustments whatsoever, that the orbit of Mercury should precess by an extra 43 seconds of arc per century should the General Theory of Relativity be correct.

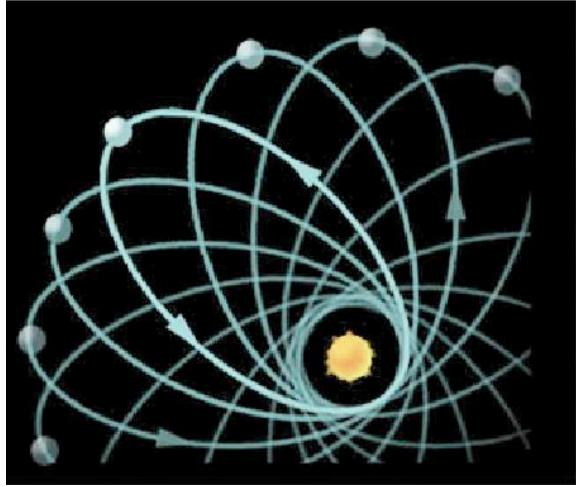


Figure 7.20: Artist's version of the precession of Mercury's orbit. Most of the effect is due to the pull from the other planets but there is a measurable effect due to the corrections to Newton's theory predicted by the General Theory of Relativity.

### 7.12.2 Gravitational red-shift.

We saw in Sec. 7.4 that light leaving a region where the gravitational force is large will be shifted towards the red (its wavelength increases; see Figs. ??,7.9); similarly, light falling into a region where the gravitational pull is larger will be shifted towards the blue. This prediction was tested in Harvard by looking at light as it fell from a tower (an experiment requiring enormous precision since the changes in the gravitational force from the top to the bottom of a tower are minute) and the results agree with the predictions from the General Theory of Relativity.

The gravitational red-shift was also tested by looking at the light from a type of stars which are very very well-studied. The observations showed that the light received on Earth was slightly redder than expected and that the reddening is also in agreement with the predictions from the General Theory of Relativity.

### 7.12.3 Light bending

If we imagine observing a beam of light in an accelerated elevator we will see that the light path is curved. By the equivalence principle the same must be true for light whenever gravitational forces are present. This was tested

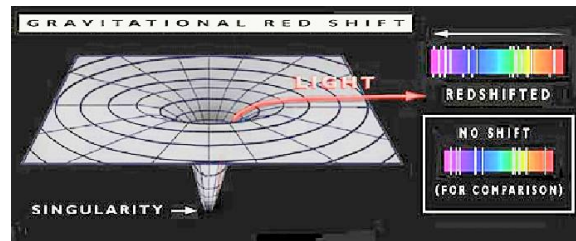


Figure 7.21: Illustration of the gravitational red-shift predicted by the General Theory of Relativity. A heavy object is denoted by a deformation of space represented by the funnel. As light leaves the vicinity of this object it is shifted towards the red: for a sufficiently compact and massive object a blue laser on the surface will be seen as red in outer space.

by carefully recording the position of stars near the rim of the sun during an eclipse (see Fig. 7.22) and then observing the same stars half a year later when there is no eclipse.

During the eclipse the observed starlight reaches us only after passing through a region where gravitational effects from the sun are very strong (that is why only stars near the rim are used), but the observation half a year later are done at a time where the gravitational effects of the sun on starlight is negligible.

It is found that the position of the stars are displaced when photographs of both situations are compared (see Fig. 7.22). The deviations are the same as the ones predicted by General Relativity. Eddington first observed this effect in 1919 during a solar eclipse. The early 20th century telegram (see Fig. ??) announcing this observation for the first time marks the change in our views about the structure of space time.

#### 7.12.4 The double pulsar

There are certain kind of stars which are called *pulsars* (see Sect. 9.3.4). These are very compact objects (they have a diameter of about 10km but are several times heavier than the sun) which emit radio pulses at very regular intervals.

In the early 80's, Taylor and Hulse (recent Nobel prize winners for this work) discovered a system where one pulsar circles another compact object. Because the pulsar pulses occur at very regular intervals, they can be used as a clock. Moreover there are several physical effects which can be used to determine the shape of the orbits of the pulsar and the compact object. It

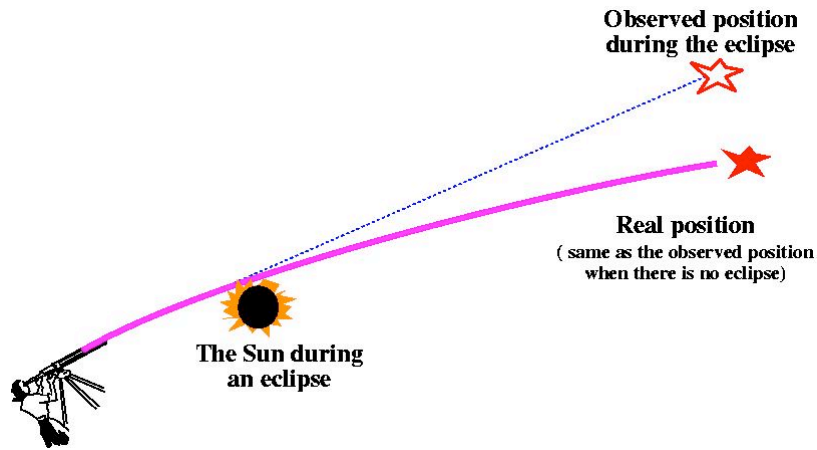


Figure 7.22: Illustration of the effects of the gravitational bending of light: during an eclipse the observed positions of the stars will be shifted away from the Sun.

was found that these objects are slowly spiraling into each other, indicating that the system is losing energy in some way.

This system can also be studied using the General Theory of Relativity which predicts that the system should radiate gravitational waves carrying energy with them and producing the observed changes. These predictions are in perfect agreement with the observations. This is the first test of General Theory of Relativity using objects outside our solar system.

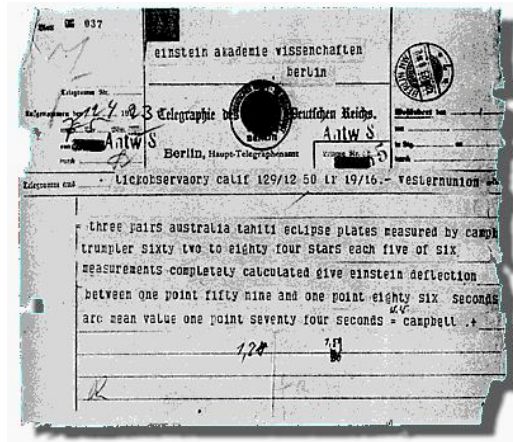


Figure 7.23: Eddington's telegram to Einstein announcing the observation of the bending of light by a gravitational force as predicted by the General Theory of Relativity.